Exercise 22

Solve the initial-value problem.

$$4y'' - 20y' + 25y = 0, \quad y(0) = 2, \quad y'(0) = -3$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad \frac{dy}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y}{dx^2} = r^2e^{rx}$$

Plug these formulas into the ODE.

$$4(r^2e^{rx}) - 20(re^{rx}) + 25(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$4r^2 - 20r + 25 = 0$$

Solve for r.

$$(2r - 5)^2 = 0$$

$$r = \left\{\frac{5}{2}\right\}$$

Two solutions to the ODE are $e^{5x/2}$ and $xe^{5x/2}$. By the principle of superposition, then,

$$y(x) = C_1e^{5x/2} + C_2xe^{5x/2}.$$

Differentiate the general solution.

$$y'(x) = \frac{5}{2}C_1e^{5x/2} + C_2e^{5x/2} + \frac{5}{2}C_2xe^{5x/2}$$

Apply the initial conditions to determine C_1 and C_2.

$$y(0) = C_1 = 2$$

$$y'(0) = \frac{5}{2}C_1 + C_2 = -3$$

Solving this system of equations yields $C_1 = 2$ and $C_2 = -8$. Therefore, the solution to the initial value problem is

$$y(x) = 2e^{5x/2} - 8xe^{5x/2}.$$

Below is a graph of $y(x)$ versus x.